A New Method of Determining Instability of Linear System

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***Abstract*— In this paper, an algorithm is presented for identification of real eigenvalues on right half of the s-plane, for linear systems, hence determining instability of the system. The proposed approach is based on Gerschgorin theorem and a new approach of Bisection method. The method is efficient since there is no need to determine all real eigenvalues and also characteristic polynomial of the system matrix. It has been found that in some class of control system problems, the method needs minor computations. The proposed approach is useful, particularly, in power system applications, where the order of system is large. A power system problem and numerical examples are illustrated using proposed algorithm.**

***Keywords*— Instability, System matrix, Gerschgorin theorem, Bisection method, Power system problem.**

I. Introduction

The stability problems arise mainly in the power system whenever perturbation occurs. These perturbations may occur because of change in the parameters of the system. The system response following the perturbation may be either stable or unstable. The control engineers are always interested in whether or not the system is stable. In practice, the systems are nonlinear in nature. The analysis and synthesis of nonlinear system are quiet difficult. So, these nonlinear systems are to be linearised around the operating point to obtain linear state variable model described by the following state variable equation,

$$\dot{x} = Ax + Bu \cdot \tag{1}$$

Where,

$A =$ System matrix, $B =$ Input matrix,
$x =$ State vector, $u =$ Control vector.

System matrix A consists of parameters of the system and its eigenvalues play very important role in the stability of the system, particularly in the power system. From eq. (1), the eigenvalues of the matrix A gives the system behavior, whether the system is stable or unstable and if stable, how much it is relatively stable. Thus, the eigenvalues are in general, functions of all control and design parameters. The stability of the system can be determined by applying Routh's criterion to the characteristic polynomial of the system [1]. This criterion gives the presence of eigenvalues of the system on the right-half of the s-plane. In the similar manner, the proposed method tries to determine the presence of real eigenvalues of the system matrix A belonging to the right half of the s-plane without computing the actual characteristics polynomial and eigenvalues. In [2], technique is presented to identify real eigenvalues using Gerschgorin theorem [3]. But, the algorithm presented in [2] fails when the eigenvalues of the system are of repetitive nature and therefore the system which is actually unstable, may show stable by the existing algorithm [2]. Today, fast computing software such as Matlab is available by which stability of any system can be tested very easily. But, in some cases, because of rounding of errors, truncation errors and ill conditioning, results shown by Matlab may be inaccurate. Hence, in such class of problems, Gerschgorin bounds will be helpful, because based on the location of the bounds, we can decide the stability of system. Moreover, the calculations of bounds require very minor computations in comparison with the calculation of actual eigenvalues.

Thus, in this paper, an algorithm is presented to check the instability of the system matrix using Gerschgorin theorem [5-7] and a new approach of Bisection method [8]. In order to show the effectiveness of the proposed method, the same power system model [4] is considered which has been considered in [2] and computational efficiency is highlighted. Similarly, various other examples are also illustrated.

In this paper, the following notations are used in mathematical developments. Complex plane is dented by $\mathbb{C}$; Open left-half plane is denoted by $\mathbb{C}^-$; open right-half plane is denoted by $\mathbb{C}^+$; belong to is denoted as $\in$.

II. GERSCHGORIN THEOREM

For $n \geq 2$, let A be any $n \times n$ real matrix (written $A = [a_{i,j}] \in \mathbb{R}^{n \times n}, i, j = 1, 2, \ldots, n$), and let $\sigma(A)$ denotes its spectrum (i.e., $\{\lambda \in \mathbb{C} : \det|A - \lambda I_n| = 0\}$). A familiar result of Gerschgorin as follows.

Theorem II.1:

The largest eigenvalue in modulus of square matrix A cannot exceed the largest sum of the module of the elements along any row or any column.

DOI: 01.IJCom.02.01.113

$$|\lambda| = \max_{i} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| \right] i = 1,2,\ldots,n \qquad (2)$$

Since the eigenvalues of A^T are same as those of A, therefore, the theorem is also true for columns.

Theorem II. 2 :

Let p_k be the sum of moduli of the elements along the k^{th} row excluding the diagonal element $a_{k,k}$. Then every eigenvalue of lies inside or on the boundary of at least one of the circles in s-plane.

$$|\lambda_i - a_{k,k}| \le \sum_{i=1, i\neq k}^{n} |a_{k,i}| = p_k, k = 1,2,\ldots,n, \qquad (3)$$

where, $a_{k,k}$ is center of circle with radius p_k. By denoting each of these n circles by R_k, we get,

$$R_k = \{|\lambda - a_{k,k}|\} = p_k, k = 1,2,\ldots,n \qquad (4)$$

Therefore, each of the eigenvalues of matrix A must lie in the union of 'S' of these n circles in the s-plane, i.e.,

$$S = \bigcup_{k=1}^{n} R_k \qquad (5)$$

It is also noted that A and A^T have the same eigenvalues, Thus by applying *theorem II.2* to A^T, it yields another set $\hat{S}$. Thus,

$$\sigma(A) = S \cap \hat{S} \qquad (6)$$

In case of real symmetric matrix A and A^T are same. Thus,

$$\sigma(A) = S \text{ or } \hat{S}. \qquad (7)$$

The eigenvalues of matrix A will lie in the intersection of these two unions S and $\hat{S}$ in the s-plane. Thus, by considering the intersection of these Gerschgorin circles, bounds under which real eigenvalues will fall, are obtained. The bounds obtained are nothing but the extreme ends of the intersection of Gerschgorin circles. Let these bounds are denoted by E and D, where D is the extreme left bound and E is the extreme right bound [6-7].

If D, E $\in \mathbb{C}^-$, then $\forall \lambda_i \in \mathbb{C}^-, i = 1,2,\ldots,n$, (all the eigenvalues lying on the left of the s-plane), system is stable.

If D, E $\in \mathbb{C}^+$, then $\forall \lambda_i \in \mathbb{C}^+, i = 1,2,\ldots,n$, (all the eigenvalues lying on the right half of the s-plane), system is unstable.

Remark 1: The above theorem is also true for the complex elements in the system matrix.

III. MAIN RESULTS

Theorem III. 1:

When there exist odd number of eigenvalues on $\mathbb{C}^+$ for a given system matrix $A \in \mathbb{R}^{n\times n}$, then,

$$|\lambda I - A|_{\lambda=0} < 0. \qquad (8)$$

Proof: Consider A as any $n \times n$ real matrix (written $A = [a_{i,j}] \in \mathbb{R}^{nxn}, i, j = 1,2,\ldots,n$), and $\lambda_i \in \mathbb{C}, i = 1,2,\ldots,n$ be the eigenvalues of the matrix. Now,

$$|\lambda I - A| = f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n). \quad (9)$$

From eq. (9), depending upon the value of λ, we get,

$$f(\lambda)_{\lambda=0} > 0, \text{ for } \forall \lambda_i \in \mathbb{C}, \mathbb{C}^{c(-,+)} i = 1,2,\ldots,n, \quad (10)$$

where, $\mathbb{C}^{c(-,+)}$ indicates only complex conjugate eigenvalues on $\mathbb{C}^-$ and $\mathbb{C}^+$.

$$f(\lambda)_{\lambda=0} > 0, \text{ for } \forall \lambda_i \in \mathbb{C}^-, \mathbb{R}, i = 1,2,\ldots,n, \quad (11)$$

and for $\forall \lambda_i \in \mathbb{C}^+, \mathbb{R}, i = 1,2,\ldots,n$, we get,

$$f(\lambda)_{\lambda=0} > 0 \text{ if } i = 2,4,6,\ldots,n, \text{ i.e., } n \text{ is even} \qquad (12)$$

and

$$f(\lambda)_{\lambda=0} < 0 \text{ if } i = 1,3,5,\ldots,n, \text{ i.e., } n \text{ is odd.} \qquad (13)$$

Thus, from above eq. (9-13), it is proved that when there exist odd number of real eigenvalues on $\mathbb{C}^+$, then,

$$|\lambda I - A|_{\lambda=0} < 0. \qquad (14)$$

Theorem III. 2:

Suppose for a given matrix,,

$A \in \mathbb{R}^{n\times n}, \Delta_1 = f(\lambda)_{\lambda=0} = |\lambda I - A|_{\lambda=0}$ and $\Delta_2 = f(\lambda)_{\lambda=e^+} = |\lambda I - A|_{\lambda=e^+}$, then there exist at least one eigenvalues $\in \mathbb{C}^+$, when $|\Delta_1| > |\Delta_2|$.

Proof: Consider a continuous function $f(\lambda)$ which include all the real and complex eigenvalues, lying on $\mathbb{C}^-$ and $\mathbb{C}^+$. It can be written as

$$f(\lambda) = f(\lambda_{rC^-}) \times f(\lambda_{rC^+}) \times f(\lambda_{cC^-}) \times f(\lambda_{cC^+}) \qquad (15)$$

DOI: 01.IJCom.02.01.113

Where,

$$f(\lambda_{r\mathbb{C}^-1}) = (\lambda - x_i)(\lambda - x_{i+1})\ldots(\lambda - x_p), \qquad (16)$$

$$f(\lambda_{r\mathbb{C}^+}) = (\lambda - x_{p+1})(\lambda - x_{p+2})\ldots(\lambda - x_r), \qquad (17)$$

$$f(\lambda_{c\mathbb{C}^-}) = ((\lambda - x_{r+1})^2 + (y_{r+1})^2)((\lambda - x_{r+3})^2 + (y_{r+3})^2)\ldots((\lambda - x_s)^2 + (y_s)^2), \qquad (18)$$

$$f(\lambda_{c\mathbb{C}^+}) = ((\lambda - x_{s+1})^2 + (y_{s+1})^2)((\lambda - x_{s+3})^2 + (y_{s+3})^2)\ldots((\lambda - x_n)^2 + (y_n)^2). \qquad (19)$$

$f(\lambda_{r\mathbb{C}^-}), f(\lambda_{r\mathbb{C}^+})$, shows the product of all the real eigenvalues on the $\mathbb{C}^-$ and $\mathbb{C}^+$ respectively. Similarly, $f(\lambda_{c\mathbb{C}^-})$ and $f(\lambda_{c\mathbb{C}^+})$ shows the product of all complex conjugate eigenvalues on the $\mathbb{C}^-$ and $\mathbb{C}^+$ respectively. Therefore, from eq.(15) and eq.(16-19), the following conclusion can be drawn.

$$\text{if } \lambda \in \mathbb{R}, \mathbb{C}^-, \text{then } f(\lambda_{r\mathbb{C}^-})_{\lambda=\varepsilon^+} > f(\lambda_{r\mathbb{C}^-})_{\lambda=0}, \qquad (20)$$

$$\text{if } \lambda \in \mathbb{C}, \mathbb{C}^-, \text{ then } f(\lambda_{c\mathbb{C}^-})_{\lambda=\varepsilon^+} > f(\lambda_{c\mathbb{C}^-})_{\lambda=0}, \qquad (21)$$

$$\text{if } \lambda \in \mathbb{R}, \mathbb{C}^+, \text{then } f(\lambda_{r\mathbb{C}^+})_{\lambda=\varepsilon^+} < f(\lambda_{r\mathbb{C}^+})_{\lambda=0}, \qquad (22)$$

$$\text{if } \lambda \in \mathbb{C}, \mathbb{C}^+, \text{then } f(\lambda_{c\mathbb{C}^+})_{\lambda=\varepsilon^+} < f(\lambda_{c\mathbb{C}^+})_{\lambda=0}. \qquad (23)$$

From above eq. (20-23), when $f(\lambda)_{\lambda=0} > f(\lambda)_{\lambda=\varepsilon}$, then there will be existence of eigenvalues either real or complex, on the $\mathbb{C}^+$, but the converse may not be true. This completes the proof.

Theorem III. 3 :

For a continuous function $f(\lambda)$, if there is a repeated (double) root at t_d, δ_x and δ_y be a small value in its neighborhood on both sides and assuming that there is no eigenvalue between δ_x and δ_y except double root at t_d

$$\text{If } (t_d - \delta_x) > (t_d - \delta_y) \text{ and } f(\lambda)_{\lambda=t_d-\delta_x} > 0, \text{ then,}$$
$$f(\lambda)_{\lambda=t_d+\delta_y} > 0 \text{ and } f(\lambda)_{\lambda=t_d-\delta_x} > f(\lambda)_{\lambda=t_d+\delta_y} \qquad (24)$$

Proof: Suppose, there is a real eigenvalue $t_1 \in \mathbb{C}^+$, δ_x and δ_y be a small value in its neighborhood, then using conventional Bisection technique,

$$\text{if } f(t_1 - \delta_x) > 0 \text{ then } f(t_1 + \delta_y) < 0, \qquad (25)$$

or

$$\text{if } f(t_1 - \delta_x) < 0 \text{ then } f(t_1 + \delta_y) > 0. \qquad (26)$$

But, when there is a repeated root at t_d and by assuming small values δ_x and δ_y in its neighborhood, the above equations are not satisfied. It is because there are two sign changes at the same moment, therefore, effectively there is no sign change (ex. For first root, - sign change to + and for the second root, – sign change to +, so, finally, two negative sign changes to positive sign and two positive sign change to positive sign). But, the value of the determinant decreases while approaching towards double root and its value becomes zero at double root and there after the value of determinant increase or decrease either in positive or negative direction, having same sign which is before crossing the root. Thus, for double root at t_d, if $(t_d - \delta_x) > (t_d - \delta_y)$, then,

$$f(t_d - \delta_x) > 0, f(t_d + \delta_y) > 0 \qquad (27)$$

and

$$f(\lambda)_{\lambda=t_d+\delta_x} > f(\lambda)_{\lambda=t_d+\delta_y}. \qquad (28)$$

This proves the theorem.

Remark 2: We have only considered the case of repeated eigenvalue when $f(t_d - \delta_x) > 0$. It is because when $f(t_d - \delta_x) < 0$, then the system is unstable.

Remark 3: While moving λ from 0 to E in suitable number of steps, whenever $|\lambda I - A|$ is negative or there is a decrease in its value at any instant, then the system is always unstable. Based on the concepts of Gerschgorin bounds and using above theorems, we are presenting algorithm for identification of real eigenvalues on the $\mathbb{C}^+$, hence the instability of the system.

IV. ALGORITHM

Using above theorems and Gerschgorin bounds, a step by step procedure for determining instability of the system matrix A is as given below.

Step 1: Enter order n and elements of the matrix, $A = [a_{ij}] \in \mathbb{R}^{n \times n}$.

Step 2: Calculate bounds D and E using Gerschgorin Theorem.

Step 3: If D, E < 0, then the system is stable, otherwise, go to next step.

Step 4: If D, E > 0, then the system is unstable, otherwise, go to next step.

Step 5: If D < 0 and E = 0, then if $\Delta_0 = |\lambda I - A|_{\lambda=0} = 0$, then system is unstable, otherwise, the system is stable.

DOI: 01.IJCom.02.01.113

If D < 0 and E > 0, or D > 0 and E < 0, then the following steps are applicable.

Step6: For $l = 0$, we calculate, $D_0 = |l\,I - A|_{l=0}$, if $D_0 < 0$, then the system is unstable, otherwise, go to next step.

Step7: For $l = e^+$, we calculate, $D_1 = |l\,I - A|_{l=e^+}$, if $D_1 < 0$ or $|D_0| > |D_1|$, then it indicates the existence of at least one eigenvalue belong to $\pounds^+$. Hence, the system is unstable, otherwise, go to next step.

Step 8: If there are sign changes in the values of $D = |l\,I - A|$ when λ is varying from zero to E in H number of steps, or there is a decrease in the value of the determinant D at any instant, or there is repeated eigenvalue as per the *theorem III.3*, then, it indicates the existence of real eigenvalues belong to . Thus, the system is unstable.

V. APPLICATIONS

V.1 Example:

Consider the matrix *A* as [7]

$$[A] = \begin{bmatrix} -5 & -1 & 0 & 0 \\ -4 & -5 & 0 & 0 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 3 & -9 \end{bmatrix}$$

Now applying Gerschgorin theorem to above matrix, we calculate bounds, as D=-12 and E=-1. Since the bounds are completely lying on the left half of the s-plane, the system is stable. The Gerschgorin circles and bounds are shown in fig. 1. The actual eigenvalues of the system matrix are -3.0000, -7.0000 -3.3944, -10.6056. The stability has been decided without computing the characteristic polynomial and eigenvalues, hence require minor computations. For such class of problems, proposed analysis is efficient in comparison with the Matlab where in eig (a) command is used to identify eigenvalues and hence the stability.

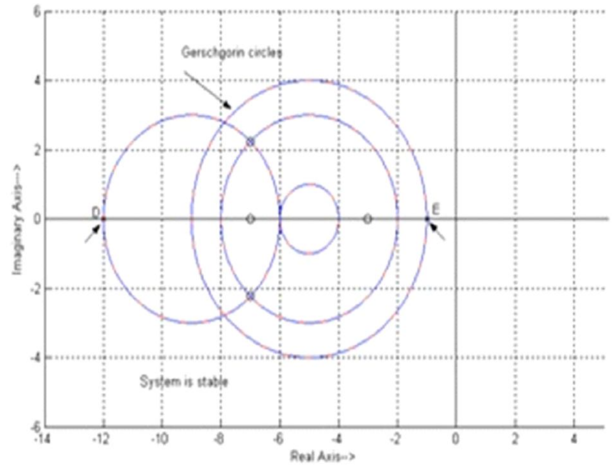

Fig.1. Gerschgorin circles and bounds for example *V.1*.

V. 2 Example

Consider the system matrix [2, 4]

$$[A] = \begin{bmatrix} -1.18 & 0 & 0 & 0.985 & 0 \\ 0 & 0 & 314 & 0 & 0 \\ -0.341 & 0 & -0.5 & 0.375 & 0.1125 \\ 48.1 & -107.5 & -205 & -54 & 431.5 \\ 295 & 295 & -170 & -638 & -19.65 \end{bmatrix}$$

Applying Gerschgorin theorem to above matrix A, we get D=-680 and E=680, as shown in fig. 2. Now, we apply proposed algorithm to the above matrix. Using step 5 of the algorithm, we get, $D_0 = |l\,I - A|_{l=0} = -6116382$. Since, the determinant is negative, the system is unstable. The actual eigenvalues of the matrix *A* are $-.7.52, -0.35, -8.5, -37.04 \pm j25$. The proposed analysis needs only one iteration, whereas as the existing method by pusadkar *et al.* [2] needs larger number of iterations.

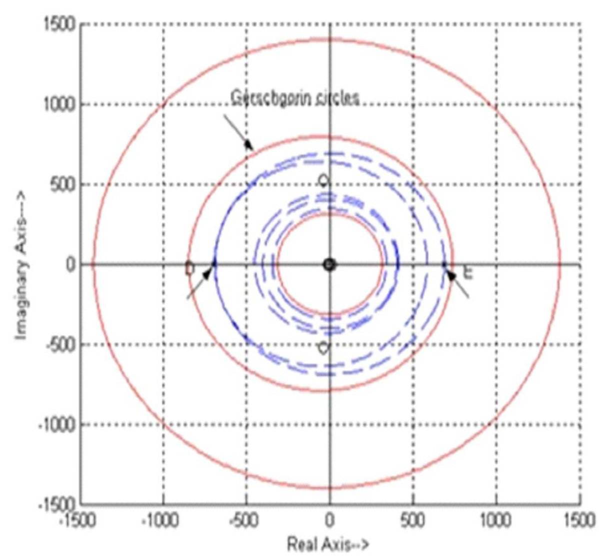

Fig. 2. Gerschgorin circles and bounds for example *V.2*.

DOI: 01.IJCom.02.01.113

V.3 Example: (Identification of repeated eigenvalues)

Consider the matrix A as

$$[A]= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -15 & 9 \end{bmatrix}$$

The Gerschgorin circles and bounds for the above matrix A are E =25 and D=-25 as shown in fig. 3.

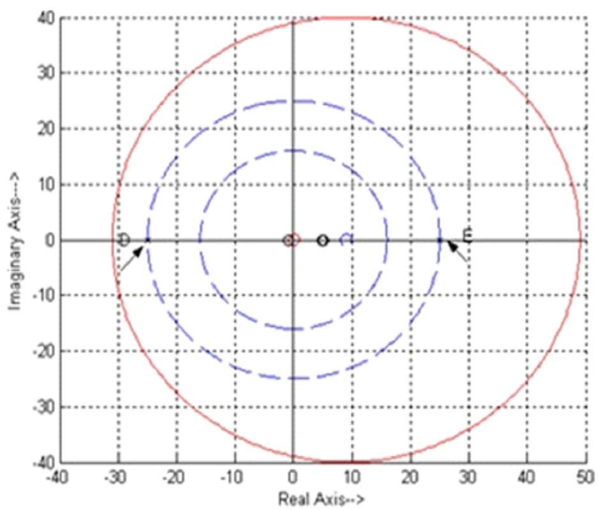

Fig. 3. Gerschgorin circles for Matrix [A]

Now we check the stability using the proposed algorithm. The $|l\,I - A|$ for various values of l are as given below in table 1.

TABLE 1: IDENTIFICATION OF REPEATED EIGENVALUES

Steps	l	$\|l\,I - A\|$
1	0	25.00
2	0.1	26.4110
3	2.08	26.22
4	4.16	3.58
5	6.24	11.13
6	8.32	102.72

It is observed from Table 1, that the value of determinant decreases from step 2 to step3. Hence, the system is unstable, and there is no need for further iterations to check stability.

But, in order to check the repeated eigenvalue, we have shown further iterations. From step 4 to step 5 in the Table 1, instead of decrease in the value of determinant, there is an increase in its value.So, from *theorem III. 3*, the value of determinant is increased after a decrease, without change in sign, hence it is concluded that there exist repeated eigenvalue between 4.16 and 6.24 on the real axis of the s-plane. This shows that the system is unstable. The approximate repeated eigenvalue is the average of 4.16 and 6.24, i.e., 5.2. The actual eigenvalues of the above matrix are -1, 5, and 5. Thus, the drawback of the paper [2] is improved in the proposed analysis.

VI. CONCLUSION

The proposed method is useful for determining the instability of the system when there exist real eigenvalues on the right half of the s-plane. It is based on the Gerschgorin theorem and a new approach of Bisection method. The method is extremely efficient when bounds E and D lie completely on either side of the s-plane or when there exist odd number of real eigenvalues lying on $\pounds^{+}$. In future, an algorithm can be developed for the identification of the complex conjugate eigenvalues which lie on . This new method will be used as an alternative to Routh Hurwitz criteria for the stability of the system.

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DOI: 01.IJCom.02.01.113